

Counting points on bilinear and trilinear hypersurfaces

Thomas Reuss

Mathematical Institute, University of Oxford

reuss@maths.ox.ac.uk

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Abstract

Consider an irreducible bilinear form $f(x_1, x_2; y_1, y_2)$ with integer coefficients. We derive an upper bound for the number of integer points $(\mathbf{x}, \mathbf{y}) \in \mathbb{P}^1 \times \mathbb{P}^1$ inside a box satisfying the equation $f = 0$. Our bound seems to be the best possible bound and the main term decreases with a larger determinant of the form f . We further discuss the case when $f(x_1, x_2; y_1, y_2; z_1, z_2)$ is an irreducible non-singular trilinear form defined on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, with integer coefficients. In this case, we examine the singularity and reducibility conditions of f . To do this, we employ the Cayley hyperdeterminant D associated to f . We then derive an upper bound for the number of integer points in boxes on such trilinear forms. The main term of the estimate improves with larger D . Our methods are based on elementary lattice results.

1 Bilinear Forms

We note the following fact about bilinear forms:

Lemma 1. *Let $A = (a_{ij})$ be a 2×2 matrix with associated bilinear form $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}$. Then f is irreducible if and only if $\det(A) \neq 0$. If f is reducible then it factorizes into a product of two linear factors.*

Proof. This is a well known result and can quickly be verified by direct calculation. \square

We will prove the following theorem:

Theorem 2. *Let $A = (a_{ij})$ be a 2×2 matrix with associated bilinear form*

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}$$

and determinant $\Delta := \det(A)$. Suppose that $\gcd(a_{11}, a_{21}, a_{12}, a_{22}) = 1$. We assume that f is irreducible over \mathbb{Z} so that $\Delta \neq 0$. Let X_1, X_2, Y_1, Y_2 be real numbers ≥ 1 and define

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) = \# \{(\mathbf{x}, \mathbf{y}) : |x_i| \leq X_i, |y_i| \leq Y_i, (x_1, x_2) = (y_1, y_2) = 1, f(\mathbf{x}, \mathbf{y}) = 0\}.$$

Then

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) \ll \min \left\{ X_1 X_2, Y_1 Y_2, d(\Delta) \left(\sqrt{\frac{X_1 X_2 Y_1 Y_2}{|\Delta|}} + 1 \right) \right\}.$$

Proof. We want to count solutions to the equation

$$f(\mathbf{x}, \mathbf{y}) = x_1(a_{11}y_1 + a_{12}y_2) + x_2(a_{21}y_1 + a_{22}y_2) = 0,$$

say. Let us write this equation as

$$f(\mathbf{x}, \mathbf{y}) = x_1 L_1(\mathbf{y}) + x_2 L_2(\mathbf{y}) = 0.$$

First let us consider the case when $L_1(\mathbf{y}) = 0$. Note that \mathbf{y} is a primitive vector and hence in particular, it is non-zero. If furthermore, $L_2(\mathbf{y}) = 0$ then we have a non-zero solution to the equation $A\mathbf{y} = 0$ which is impossible since $\Delta \neq 0$. Thus, we have $x_2 = 0$ which implies that $x_1 = \pm 1$ since \mathbf{x} is a primitive vector. There are at most 4 choices for \mathbf{y} such that $L_1(\mathbf{y}) = 0$. This is easy to see after we divide a_{11} and a_{12} by (a_{11}, a_{12}) in the equation $a_{11}y_1 + a_{12}y_2 = 0$. Thus, the case $L_1(\mathbf{y}) = 0$ contributes $O(1)$ to $\mathcal{N}(\mathbf{X}, \mathbf{Y})$. The case $L_2(\mathbf{y}) = 0$ is analogous. Now, if $x_1 = 0$ then $x_2 = \pm 1$ which reduces to the case $L_2(\mathbf{y}) = 0$. Thus, we may assume that

$$x_1 x_2 L_1(\mathbf{y}) L_2(\mathbf{y}) \neq 0.$$

Since x_1 and x_2 are coprime, we can therefore deduce the existence of a non-zero integer q such that

$$\begin{aligned} -qx_1 &= a_{21}y_1 + a_{22}y_2, \\ qx_2 &= a_{11}y_1 + a_{12}y_2. \end{aligned}$$

If we set

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then we have $qP\mathbf{x} = A\mathbf{y}$. Interchanging the roles of \mathbf{x} and \mathbf{y} , we can similarly deduce that there is a non-zero integer q' such that $q'P\mathbf{y} = A^T\mathbf{x}$. Combining the two equations, we get

$$qq'\mathbf{x} = -\Delta\mathbf{x}.$$

The vector \mathbf{x} is primitive and therefore non-zero. This allows us to conclude that

$$qq' = -\Delta.$$

Since Δ is non-zero, there are $d(\Delta)$ choices for q . We now fix one such choice and count the respective contribution to $\mathcal{N}(\mathbf{X}, \mathbf{Y})$. We first note that $q \mid A\mathbf{y}$. This will give us a lattice condition on \mathbf{y} , which is described in the following lemma:

Lemma 3. *Fix an integer $m \geq 2$. Let $M = (m_{ij})$ be an $m \times 2$ matrix. Let q be a non-zero integer such that q divides all of the $m(m-1)/2$ minors of size 2×2 of M .*

Furthermore, assume that there exists no prime $p \mid q$ which divides all the entries of M . Then

$$\Lambda_q = \{\mathbf{v} \in \mathbb{Z}^2 : q \mid M\mathbf{v}\}$$

is a lattice of determinant q .

Proof. (of Lemma) It is clear that Λ_q is an integer lattice. We proceed to calculate its determinant. We decompose $q = \prod p_i^{e_i}$ into its prime powers and consider the (additive) group homomorphism

$$\phi_i : \mathbb{Z}^2 \rightarrow \mathbb{Z}^m / p_i^{e_i} \mathbb{Z}^m$$

given by $\phi_i(\mathbf{v}) = M\mathbf{v} \pmod{p_i^{e_i}}$. By assumption of the lemma, there exists an element of M which is not divisible by p_i . We assume without loss of generality that $p_i \nmid m_{11}$. The other cases are analogous. Let m_{11}^{-1} be the multiplicative inverse of m_{11} modulo $p_i^{e_i}$. Let $\mathbf{w} \equiv m_{11}^{-1}(m_{11}, m_{21}, \dots, m_{m1})^T \pmod{p_i^{e_i}}$. We claim that $\text{Im}(\phi_i)$ is cyclic with generator \mathbf{w} . First note that for any $\lambda \in \mathbb{Z}/p_i^{e_i}\mathbb{Z}$, $\phi_i(\lambda m_{11}^{-1}, 0) \equiv \lambda \mathbf{w} \pmod{p_i^{e_i}}$. Furthermore, assume $\mathbf{u} = M\mathbf{v}$. Then for $j = 2, \dots, m$:

$$u_1 m_{j1} m_{11}^{-1} = m_{j1} v_1 + m_{12} m_{j1} m_{11}^{-1} v_2 \equiv m_{j1} v_1 + m_{j2} v_2 = u_j,$$

where the last equality follows from the fact that $p_i^{e_i}$ divides the minor $m_{11}m_{i2} - m_{12}m_{i1}$. Hence, we have indeed that $\mathbf{u} = u_1 \mathbf{w}$, and \mathbf{w} generates the image of ϕ_i . We now consider the group homomorphism

$$\phi : \mathbb{Z}^2 \rightarrow \mathbb{Z}^m / q \mathbb{Z}^m$$

given by $\phi(\mathbf{v}) = M\mathbf{v} \pmod{q}$. By the Chinese Remainder Theorem and by what we just showed, we have that $|\text{Im}(\phi)| = q$. We note that $\text{Ker}(\phi) = \Lambda_q$ and it follows by the first Isomorphism Theorem for groups that

$$\det(\Lambda_q) = [\mathbb{Z}^2 : \Lambda_q] = |\text{Im}(\phi)| = q,$$

which proves the lemma. □

We now recall that $q \mid A\mathbf{y}$. Thus, Lemma 3 shows that $\mathbf{y} \in \Lambda_y$, where Λ_y is a lattice of determinant $|q|$. The bounds $|y_1| \leq Y_1$ and $|y_2| \leq Y_2$ let us deduce that \mathbf{y} is inside the ellipse defined by

$$E_y = \left\{ \mathbf{y} \in \mathbb{R}^2 : \left(\frac{y_1}{\sqrt{2}Y_1} \right)^2 + \left(\frac{y_2}{\sqrt{2}Y_2} \right)^2 \leq 1 \right\}.$$

The area of this ellipse is $A(E_y) = 2\pi Y_1 Y_2$. Similarly, $\mathbf{x} \in E_x \cap \Lambda_x$, where E_x is an ellipse of area $A(E_x) = 2\pi X_1 X_2$, and Λ_x is an integer lattice of determinant $|q'|$. Thus, we have reduced the problem to one where we need to count primitive lattice points inside an ellipse. We will employ Lemma 2 of Heath-Brown [2], which we state here for convenience:

Lemma 4. *Let $\Lambda \subseteq \mathbb{R}^2$ be a lattice of determinant $\det(\Lambda)$. Let $E \subseteq \mathbb{R}^2$ be an ellipse, centered at the origin, together with its interior, and let $A(E)$ be the area of E . Then there is a positive number $\alpha = \alpha(\Lambda, E)$ and a basis $\mathbf{b}_1, \mathbf{b}_2$ of Λ such that $\lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 \in E$ implies $|\lambda_1| \leq \alpha$ and $|\lambda_2| \leq A(E)/(\alpha \det(\Lambda))$. Furthermore, the number of primitive lattice points in Λ contained in E is at most*

$$4 \left(\frac{A(E)}{\det(\Lambda)} + 1 \right).$$

Thus,

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) \ll \sum_{qq'=-\Delta} \min \left\{ \frac{X_1 X_2}{|q'|}, \frac{Y_1 Y_2}{|q|} \right\} + d(\Delta)$$

Since $qq' = -\Delta$, the worst case for q is when $q^2 = |\Delta| Y_1 Y_2 / (X_1 X_2)$. Thus, we get the bound

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) \ll d(\Delta) \left(\sqrt{\frac{X_1 X_2 Y_1 Y_2}{|\Delta|}} + 1 \right).$$

The bound

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) \ll \min(X_1 X_2, Y_1, Y_2)$$

can be deduced as follows. There are $O(X_1 X_2)$ choices for \mathbf{x} . Fix one such \mathbf{x} . The equation $f(\mathbf{x}, \mathbf{y}) = 0$ cannot be zero identically because \mathbf{x} is a primitive vector and $\Delta \neq 0$. Thus, there are $O(1)$ choices for \mathbf{y} . A similar argument yields the bound $O(Y_1 Y_2)$. This proves the theorem. \square

2 Trilinear Forms

Let $A = (a_{ijk})$ be a $2 \times 2 \times 2$ hypermatrix. We associate with A its hyperdeterminant

$$\begin{aligned} D := & a_{122}^2 a_{211}^2 + a_{111}^2 a_{222}^2 + a_{212}^2 a_{121}^2 + a_{112}^2 a_{221}^2 \\ & - 2a_{111}a_{122}a_{211}a_{222} - 2a_{211}a_{122}a_{112}a_{221} - 2a_{211}a_{122}a_{212}a_{121} - 2a_{222}a_{111}a_{212}a_{121} \\ & - 2a_{222}a_{111}a_{112}a_{221} - 2a_{112}a_{121}a_{212}a_{221} + 4a_{112}a_{121}a_{211}a_{222} + 4a_{111}a_{122}a_{212}a_{221}. \end{aligned}$$

We will give a brief outline of some properties of D below. For some further details on hyperdeterminants, the reader may consult the text by Gel'fand, Kapranov and Zelevinsky [1]. We define a trilinear form $f = f_A : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}^T (A\mathbf{z})\mathbf{y}.$$

Here, $A\mathbf{z}$ denotes the standard hypermatrix-vector multiplication along the third dimension of A . In particular,

$$A\mathbf{z} = M_{xy}(\mathbf{z}) := z_1 \begin{pmatrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{pmatrix} + z_2 \begin{pmatrix} a_{112} & a_{122} \\ a_{212} & a_{222} \end{pmatrix}$$

is an ordinary 2x2 matrix depending only on \mathbf{z} . We similarly define 2x2 matrixes $M_{yz}(\mathbf{x})$ and $M_{zx}(\mathbf{y})$ by

$$M_{yz}(\mathbf{x}) := x_1 \begin{pmatrix} a_{122} & a_{121} \\ a_{112} & a_{111} \end{pmatrix} + x_2 \begin{pmatrix} a_{222} & a_{221} \\ a_{212} & a_{211} \end{pmatrix}$$

and

$$M_{xz}(\mathbf{y}) := y_1 \begin{pmatrix} a_{111} & a_{112} \\ a_{211} & a_{212} \end{pmatrix} + y_2 \begin{pmatrix} a_{121} & a_{122} \\ a_{221} & a_{222} \end{pmatrix},$$

so that

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}^T M_{xy}(\mathbf{z})\mathbf{y} = \mathbf{y}^T M_{yz}(\mathbf{x})\mathbf{z} = \mathbf{z}^T M_{zx}(\mathbf{y})\mathbf{x}.$$

By taking transposes on the equation $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$ it makes sense to define

$$M_{yx}(\mathbf{z}) := M_{xy}(\mathbf{z})^T, \quad M_{zy}(\mathbf{x}) := M_{yz}(\mathbf{x})^T, \quad M_{zx}(\mathbf{y}) := M_{zx}(\mathbf{y})^T.$$

Furthermore, let

$$\begin{aligned} \Delta_{xy} = \Delta_{xy}(\mathbf{z}) &:= \det(M_{xy}(\mathbf{z})), \\ \Delta_{yz} = \Delta_{yz}(\mathbf{x}) &:= \det(M_{yz}(\mathbf{x})), \\ \Delta_{zx} = \Delta_{zx}(\mathbf{y}) &:= \det(M_{zx}(\mathbf{y})). \end{aligned}$$

Then

$$\begin{aligned}
\Delta_{xy}(\mathbf{z}) &= (a_{111}a_{221} - a_{121}a_{211})z_1^2 + (a_{111}a_{222} + a_{112}a_{221} - a_{212}a_{121} - a_{211}a_{122})z_2z_1 \\
&\quad + (a_{112}a_{222} - a_{122}a_{212})z_2^2, \\
\Delta_{yz}(\mathbf{x}) &= (a_{111}a_{122} - a_{112}a_{121})x_1^2 + (a_{111}a_{222} + a_{211}a_{122} - a_{112}a_{221} - a_{212}a_{121})x_2x_1 \\
&\quad + (a_{211}a_{222} - a_{212}a_{221})x_2^2, \\
\Delta_{zx}(\mathbf{y}) &= (a_{111}a_{212} - a_{112}a_{211})y_1^2 + (a_{111}a_{222} + a_{212}a_{121} - a_{112}a_{221} - a_{211}a_{122})y_2y_1 \\
&\quad + (a_{121}a_{222} - a_{122}a_{221})y_2^2.
\end{aligned}$$

Lemma 5. *The discriminants of the quadratic forms Δ_{xy} , Δ_{yz} and Δ_{zx} are all equal to D .*

Proof. This can be verified by direct calculation. It suffices to show that

$$\begin{aligned}
D &= (a_{111}a_{222} + a_{112}a_{221} - a_{212}a_{121} - a_{211}a_{122})^2 \\
&\quad - 4(a_{111}a_{221} - a_{121}a_{211})(a_{112}a_{222} - a_{122}a_{212}) \\
&= (a_{111}a_{222} + a_{211}a_{122} - a_{112}a_{221} - a_{212}a_{121})^2 \\
&\quad - 4(a_{111}a_{122} - a_{112}a_{121})(a_{211}a_{222} - a_{212}a_{221}) \\
&= (a_{111}a_{222} + a_{212}a_{121} - a_{112}a_{221} - a_{211}a_{122})^2 \\
&\quad - 4(a_{111}a_{212} - a_{112}a_{211})(a_{121}a_{222} - a_{122}a_{221}).
\end{aligned}$$

□

Lemma 6. *Assume that*

$$Q(\mathbf{x}) = ax_1^2 + bx_1x_2 + cx_2^2 \in \mathbb{Z}[\mathbf{x}]$$

is a binary quadratic form with discriminant $D(Q) = b^2 - 4ac = 0$. Then, $D(Q) = 0$ if and only if $Q(\mathbf{x}) = CL(\mathbf{x})^2$ for some (possibly zero) $C \in \mathbb{Q}$, and a linear form $L(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ with integer coefficients.

Lemma 7.

- i) *Assume that Δ_{xy} vanishes identically. Then Δ_{yz} or Δ_{zx} also vanishes identically.*
- ii) *If Δ_{xy} and Δ_{yz} vanish identically then f factorizes over \mathbb{Q} as*

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = L(\mathbf{y})B(\mathbf{x}, \mathbf{z}), \tag{1}$$

where L is a linear form, and B is a bilinear form. Furthermore, $\Delta_{zx}(\mathbf{y}) = \det(B)L(\mathbf{y})^2$, where $\det(B)$ is the determinant of the matrix associated to B .

- iii) *If f has a linear factor $L(\mathbf{y})$ over \mathbb{Z} then Δ_{xy} and Δ_{yz} both vanish identically.*
- iv) *f splits in three linear factors over \mathbb{Z} if and only if Δ_{xy} , Δ_{yz} and Δ_{zx} all vanish identically.*

Proof. We prove the first two claims when $a_{111}a_{222} \neq 0$. Assume that Δ_{xy} vanishes identically. Then all the coefficients of Δ_{xy} vanish, that is:

$$\begin{aligned} a_{111}a_{221} - a_{121}a_{211} &= 0 \\ a_{112}a_{222} - a_{122}a_{212} &= 0 \\ a_{111}a_{222} + a_{112}a_{221} - a_{212}a_{121} - a_{211}a_{122} &= 0. \end{aligned} \tag{2}$$

Using just the first two equations, we get

$$\Delta_{yz}(\mathbf{x}) = \frac{(a_{111}a_{222} - a_{212}a_{121})(x_1a_{122} + x_2a_{222})(x_1a_{111} + x_2a_{211})}{a_{111}a_{222}}$$

and

$$\Delta_{zx}(\mathbf{y}) = \frac{(a_{111}a_{222} - a_{211}a_{122})(y_1a_{212} + y_2a_{222})(y_1a_{111} + y_2a_{121})}{a_{111}a_{222}}.$$

If we substitute the first two equations of (2) into the third one, we obtain that

$$\frac{(a_{111}a_{222} - a_{211}a_{122})(a_{111}a_{222} - a_{212}a_{121})}{a_{111}a_{222}} = 0.$$

If $a_{111}a_{222} - a_{211}a_{122} = 0$ then $\Delta_{zx}(\mathbf{y})$ vanishes identically and

$$\Delta_{yz}(\mathbf{x}) = \frac{(a_{212}a_{121} - a_{211}a_{122})}{a_{222}a_{122}}(x_1a_{122} + x_2a_{222})^2.$$

Furthermore, $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ factorizes as follows:

$$\frac{(x_1a_{122} + x_2a_{222})(a_{122}a_{212}z_2y_1 + a_{122}y_1z_1a_{211} + a_{122}y_2z_2a_{222} + y_2z_1a_{121}a_{222})}{a_{222}a_{122}}.$$

This proves i) and ii) in the case under consideration. In the case $a_{111}a_{222} - a_{212}a_{121} = 0$, we note that $\Delta_{yz}(\mathbf{x})$ vanishes identically and the calculations are similar. It is not difficult to verify i) and ii) if $a_{111}a_{222} = 0$.

We now prove iii). If

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (b_1y_1 + b_2y_2)(c_1x_1z_1 + c_2x_1z_2 + c_3x_2z_1 + c_4x_2z_2)$$

then

$$\begin{aligned} a_{111} &= b_1c_1, \quad a_{112} = b_1c_2, \quad a_{121} = b_2c_1, \quad a_{122} = b_2c_2, \\ a_{211} &= b_1c_3, \quad a_{221} = b_2c_3, \quad a_{212} = b_1c_4, \quad a_{222} = b_2c_4, \end{aligned}$$

and it can easily be verified by direct calculation that Δ_{xy} and Δ_{yz} both vanish identically. The statement iv) follows directly from ii) and iii). \square

Lemma 8. *The following are equivalent*

i) $D = 0$.

ii) *There exists a non-trivial point in $(\mathbb{P}^1)^3$ at which all partial derivatives of f vanish.*

iii) *f is singular in $(\mathbb{P}^1)^3$.*

Proof.

ii) \Rightarrow i) Assume for a contradiction that all partial derivatives of f vanish at a point in $(\mathbb{P}^1)^3$ and that $D \neq 0$. Thus, by Lemma 6, there are precisely two distinct solutions $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{P}$ to the equation $\Delta_{xy}(\mathbf{z}) = 0$. If $M_{xy}(\mathbf{z}_1)$ is identically zero, then

$$M_{xz}(\mathbf{y})\mathbf{z}_1 = M_{xy}(\mathbf{z}_1)\mathbf{y} = \mathbf{0},$$

for all $\mathbf{y} \in \mathbb{P}$. This means that for all $\mathbf{y} \in \mathbb{P}$, there is a non-zero vector in the kernel of $M_{xz}(\mathbf{y})$. Thus, $\Delta_{xz}(\mathbf{y})$ vanishes identically and therefore $D = 0$, which contradicts our assumption. Therefore, $M_{xy}(\mathbf{z}_1)$ cannot vanish identically and there must be $\mathbf{y}_2 \in \mathbb{P}$ such that $M_{xy}(\mathbf{z}_1)\mathbf{y}_2 = 0$. We note that \mathbf{y}_2 is unique (up to a scalar multiple), since otherwise $M_{xy}(\mathbf{z}_1)$ would vanish identically. Similarly, there exists a unique $\mathbf{y}_1 \in \mathbb{P}$ such that $M_{xy}(\mathbf{z}_2)\mathbf{y}_1 = 0$. If $\mathbf{y}_1 = \mathbf{y}_2$ then

$$M_{xz}(\mathbf{y}_1)\mathbf{z}_1 = M_{xz}(\mathbf{y}_1)\mathbf{z}_2.$$

But similarly to the above argument, there must be a unique $\mathbf{z} \in \mathbb{P}$ such that $M_{xz}(\mathbf{y}_1)\mathbf{z} = 0$. But, by assumption we have that $\mathbf{z}_1 \neq \mathbf{z}_2$. Therefore, $\mathbf{y}_1 \neq \mathbf{y}_2$. Similarly, we can find $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{P}$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ such that $M_{yx}(\mathbf{z}_1)\mathbf{x}_2 = 0$ and $M_{yx}(\mathbf{z}_2)\mathbf{x}_1 = 0$. We can now write an arbitrary $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{P}^1)^3$ as

$$(a_1\mathbf{x}_1 + a_2\mathbf{x}_2, b_1\mathbf{y}_1 + b_2\mathbf{y}_2, c_1\mathbf{z}_1 + c_2\mathbf{z}_2),$$

say. This is possible since $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{y}_1, \mathbf{y}_2$ and $\mathbf{z}_1, \mathbf{z}_2$ are all bases for \mathbb{P}^1 . We can then see that

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = a_1b_1c_1f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1) + a_2b_2c_2f(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2).$$

Now we observe that $\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$. Since \mathbf{x}_1 and \mathbf{x}_2 are linearly independent, we can invert the transformation and get that $a_1 = L_1(\mathbf{x})$ and $a_2 = L'_1(\mathbf{x})$, say, where L_1 and L'_1 are linear forms with coefficients in \mathbb{Q} , depending on \mathbf{x}_1 and \mathbf{x}_2 . We find similar expressions for b_1, b_2 and c_1, c_2 to deduce that

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = L_1(\mathbf{x})L_2(\mathbf{y})L_3(\mathbf{z}) + L'_1(\mathbf{x})L'_2(\mathbf{y})L'_3(\mathbf{z}).$$

From this expression we can see directly that if f has a singular point (X, Y, Z) then it must have a linear factor, without loss of generality, $L(\mathbf{x})$, say. This linear factor must vanish at a point $\mathbf{p} \in \mathbb{P}$. We would then have for all $\mathbf{y}, \mathbf{z} \in \mathbb{P}$ that

$$0 = f(\mathbf{p}, \mathbf{y}, \mathbf{z}) = \mathbf{y}^T M_{yz}(\mathbf{p})\mathbf{z}.$$

This then gives that $M_{yz}(\mathbf{p}) = 0$ which implies as above that $D = 0$.

- i) \Rightarrow ii) Conversely, assume that $D = 0$. By Lemma 6, $\Delta_{xy}(\mathbf{z}) = cL(\mathbf{z})^2$. We first consider the case when $c = 0$. In this case f factorizes by Lemma 7. We assume without loss of generality that f splits as in (1). But in this case the gradient of f is zero when $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is picked such that $L(\mathbf{y})$ and $B(\mathbf{x}, \mathbf{z})$ simultaneously vanish. By this argument, we may assume that neither of the three determinants $\Delta_{xy}(\mathbf{z})$, $\Delta_{yz}(\mathbf{x})$, or $\Delta_{zx}(\mathbf{y})$ vanishes identically. Next, we pick a primitive \mathbf{z} such that $\Delta_{xy}(\mathbf{z}) = 0$. Similarly, we pick primitive \mathbf{x} and \mathbf{y} such that $\Delta_{yz}(\mathbf{x}) = 0$ and $\Delta_{zx}(\mathbf{y}) = 0$. This implies that there are primitive vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{z}_1, \mathbf{z}_2$ such that

$$\begin{aligned}\mathbf{x}_1^T M_{xy}(\mathbf{z}) &= M_{xy}(\mathbf{z})\mathbf{y}_1 = 0, \\ \mathbf{y}_2^T M_{yz}(\mathbf{x}) &= M_{yz}(\mathbf{x})\mathbf{z}_1 = 0, \\ \mathbf{z}_2^T M_{zx}(\mathbf{y}) &= M_{zx}(\mathbf{y})\mathbf{x}_2 = 0.\end{aligned}$$

Note that $\mathbf{x}_1^T M_{xy}(\mathbf{z}) = \mathbf{z}^T M_{zy}(\mathbf{x}_1) = 0$. Thus, $\Delta_{yz}(\mathbf{x}_1) = 0$. We recall that the discriminant Δ_{yz} is $D = 0$ and that Δ_{yz} does not vanish identically. Therefore, $\mathbf{x}_1 = \mathbf{x}$. By using the same idea, we can show

$$\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}, \quad \mathbf{y}_1 = \mathbf{y}_2 = \mathbf{y}, \quad \mathbf{z}_1 = \mathbf{z}_2 = \mathbf{z},$$

which proves that all partial derivatives of f are simultaneously zero at the non-trivial point $(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

- ii) \Leftrightarrow iii) The fact that ii) \Rightarrow iii) follows from Euler's theorem for homogeneous polynomials. It states in particular that for a homogeneous function $g(x_1, \dots, x_d)$ of order n :

$$\sum_{i=1}^d x_i \frac{\partial g}{\partial x_i} = ng(\mathbf{x}).$$

In our case, we may apply this result with $g = f$, $d = 6$ and $n = 3$. It is then clear that ii) \Rightarrow iii). The converse is trivial.

□

Let $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ be real numbers ≥ 1 . Our goal is to find an upper bound for the quantity

$$\begin{aligned}\mathcal{N}'(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) &= \#\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) : |x_i| \leq X_i, |y_i| \leq Y_i, |z_i| \leq Z_i, \\ &\quad (x_1, x_2) = (y_1, y_2) = (z_1, z_2) = 1, f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0\}.\end{aligned}$$

By Theorem 2, it suffices to study the case when f is irreducible. We note that if f factorizes then $D = 0$ but the converse may not necessarily be true as the family of examples

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = y_1 x_1 z_1 a_{111} + y_1 x_2 z_1 a_{211} + y_1 x_2 z_2 + y_2 x_2 z_1$$

shows. Here, $D = 0$ but f is irreducible if $a_{111} \neq 0$. Let

$$s = s(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \Delta_{xy}(\mathbf{z})\Delta_{yz}(\mathbf{x})\Delta_{zx}(\mathbf{y}).$$

It turns out that the number of points counted by \mathcal{N}' for which $s = 0$ may be large, even if $D \neq 0$. For example, consider

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = y_1x_1z_1a_{111} + y_1x_2z_2 + y_2x_2z_1 + y_2x_2z_2$$

for $a_{111} \neq 0$. Then, f is irreducible, $D = a_{111}^2 \neq 0$ and

$$s(\mathbf{x}, \mathbf{y}, \mathbf{z}) = a_{111}^2x_2y_1z_1(z_1 + z_2)(y_1 + y_2)(a_{111}x_1 - x_2).$$

Furthermore, it is easy to see that the number of points contributing to \mathcal{N}' for which $s = 0$ is $\gg X_1X_2 + Y_1Y_2 + Z_1Z_2$. Lemma 12 shows that this is in general also the best lower bound. We therefore exclude points for which $s = 0$ in the remaining argument.

We now state the main theorem of this section:

Theorem 9. *Let $A = (a_{ijk})$ be a $2 \times 2 \times 2$ matrix with associated trilinear form $f = f_A$ and hyperdeterminant $D := \det(A)$. We assume that f is irreducible over \mathbb{Z} and that $D \neq 0$. Let $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ be real numbers ≥ 1 and define*

$$\begin{aligned} \mathcal{N}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) &= \#\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) : |x_i| \leq X_i, |y_i| \leq Y_i, |z_i| \leq Z_i, \\ &\quad (x_1, x_2) = (y_1, y_2) = (z_1, z_2) = 1, f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0, s(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq 0\}. \end{aligned}$$

Let $T = \|f\|X_1X_2Y_1Y_2Z_1Z_2$. Then

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll T^\epsilon \left(\frac{\sqrt{X_1X_2Y_1Y_2Z_1Z_2}}{D^{1/4}} + \sqrt{X_1X_2Y_1Y_2} + Z_1Z_2 \right).$$

We note in particular, that the condition $D \geq 1$ yields the estimate

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll T^\epsilon \sqrt{X_1X_2Y_1Y_2Z_1Z_2}.$$

This can easily be obtained by permuting \mathbf{x} , \mathbf{y} and \mathbf{z} in the theorem and then taking the minimum of the resulting estimates.

We define for any primitive vector \mathbf{z} :

$$f_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) := f(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

With this notation, we think of $f_{\mathbf{z}}$ as a bilinear form in the variables \mathbf{x}, \mathbf{y} . Let the coefficients of $f_{\mathbf{z}}$ be the linear forms $L_z^{(i)} = L_z^{(i)}(\mathbf{z})$, where $i = 1, \dots, 8$. We may think of $L_z^{(i)}$ as a 1×2 row vector, the elements of $L_z^{(i)}$ being the coefficients of the form $L_z^{(i)}(\mathbf{z})$. Let $i, j \in \{1, \dots, 8\}$. We consider a 2×2 matrix which has $L_z^{(i)}$ as its first row and $L_z^{(j)}$ as its second row. Let $D_z^{(i,j)}$ be the absolute value of the determinant of this matrix. We define analogously $f_{\mathbf{y}}, L_y^{(i)}, D_y^{(i,j)}$ and $f_{\mathbf{x}}, L_x^{(i)}, D_x^{(i,j)}$.

Lemma 10. *Let q be a non-zero integer. Assume that there exists a primitive vector \mathbf{z} such that $q \mid L_z^{(i)}(\mathbf{z})$ for all i . Then $q^2 \mid D$.*

Proof. Let

$$g_z = \gcd_{i,j} D_z^{(i,j)}.$$

Our first aim is to prove that $q \mid g_z$. We fix i and j and want to show that $q \mid D_z^{(i,j)}$. If $D_z^{(i,j)} = 0$ then we are done. Thus, we assume that $D_z^{(i,j)} \neq 0$. Let $L_z^{(i)}(\mathbf{z}) = az_1 + bz_2$ and $L_z^{(j)}(\mathbf{z}) = cz_1 + dz_2$ so that $D_z^{(i,j)} = ad - bc$. There exists integers k_1 and k_2 such that

$$az_1 + bz_2 = k_1q \quad \text{and} \quad cz_1 + dz_2 = k_2q.$$

So that

$$\begin{aligned} (bc - ad)z_2 &= q(ck_1 - ak_2), \\ (bc - ad)z_1 &= -q(dk_1 - bk_2) \end{aligned}$$

We observe that $q \mid (bc - ad)z_1$ and that $q \mid (bc - ad)z_2$. Since z_1 and z_2 are coprime, we therefore get that $q \mid (ad - bc)(z_1, z_2) = ad - bc$. This proves that $q \mid D_z^{(i,j)}$ and since i and j were arbitrary, we may deduce that $q \mid g_z$. It can be shown by direct calculations that g_z is the greatest common factor of the following six 2x2 determinants:

$$\begin{array}{lll} a_{111}a_{122} - a_{112}a_{121}, & a_{111}a_{212} - a_{112}a_{211}, & a_{111}a_{222} - a_{112}a_{221}, \\ a_{212}a_{121} - a_{211}a_{122}, & a_{121}a_{222} - a_{122}a_{221}, & a_{211}a_{222} - a_{212}a_{221} \end{array}$$

From this it follows at once that $q^2 \mid D$. □

Lemma 11. *Assume that there exists a primitive vector \mathbf{z} such that $L_z^{(i)}(\mathbf{z}) = 0$ for all i . Then f factorizes or vanishes identically.*

Proof. Let $L_z^{(i)}(\mathbf{z}) = a_iz_1 + b_iz_2$ ($i = 1, \dots, 8$), say and define $a'_i = a_i/(a_i, b_i)$ and $b'_i = b_i/(a_i, b_i)$ in the case when $(a_i, b_i) \neq 0$. We recall that z_1 and z_2 are co-prime. We assume that all $L_z^{(i)}$ vanish simultaneously for the same \mathbf{z} . If $z_1 = 0$ then $z_2 = \pm 1$ and therefore $b'_i = 0$ and $a'_i = \pm 1$ for all i . This implies that $b_i = 0$ for all i , which means that f has a linear factor z_1 and the conclusion in the lemma is valid. Thus we may assume that $z_1 \neq 0$ and similarly $z_2 \neq 0$. If $z_1z_2 \neq 0$ and $L_z^{(i)}(\mathbf{z}) = 0$ then either $a_i = b_i = 0$ or $\mathbf{z} = \pm(b'_i, -a'_i)$. This implies that either all $L_z^{(i)}$ are identically 0 or that all the $L_z^{(i)}$ are pairwise proportional. In the first case, f vanishes identically and in the second case, f has a linear factor. This proves the lemma. □

Lemma 12. *Assuming that f is irreducible over \mathbb{Z} , the number of points $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ counted by \mathcal{N} for which $\Delta_{xy}(\mathbf{z}) = 0$ is $O(X_1X_2 + Y_1Y_2)$.*

Proof. We first note that Δ_{xy} cannot vanish identically by Lemma 7. Thus, there are $O(1)$ choices for \mathbf{z} . We fix one such \mathbf{z} and consider the equation $f_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) = 0$ as a bilinear equation with coefficients $L_z^{(i)}(\mathbf{z}) = a_i z_1 + b_i z_2$ ($i = 1, \dots, 8$), say. By Lemma 11, the function $f_{\mathbf{z}}(\mathbf{x}, \mathbf{y})$ cannot vanish identically. The equation $f_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) = 0$ is therefore saying that a non-zero bilinear form vanishes. This bilinear form will factorize as a product of linear forms $f_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) = L_{1,\mathbf{z}}(\mathbf{x})L_{2,\mathbf{z}}(\mathbf{y})$ because $\Delta_{xy}(\mathbf{z}) = 0$. It is clear that the equation $L_{1,\mathbf{z}}(\mathbf{x})L_{2,\mathbf{z}}(\mathbf{y}) = 0$ has $\ll X_1 X_2 + Y_1 Y_2$ solutions. \square

Fix a primitive integer vector \mathbf{z} . We want to apply Theorem 2 for bilinear equations to $f_{\mathbf{z}}$. By assumption, $\Delta(f_{\mathbf{z}}) \neq 0$. If there exists an integer q dividing all $L_z^{(i)}$ then $q^2 \mid D$ by Lemma 10. Let $L'_i = L_z^{(i)}/q$. We divide the equation $f_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) = 0$ by q so that we get $f'(\mathbf{x}, \mathbf{y}) = 0$, where f' is a bilinear form with coprime coefficients and non-zero determinant $\Delta(f') = \Delta_{xy}(\mathbf{z})/q^2$. Thus, we may deduce that:

$$\mathcal{N} : \ll \sum_{q^2 \mid D} \sum_{\substack{\mathbf{z}: |z_i| \leq Z_i \\ (z_1, z_2) = 1 \\ q \mid L_z^{(j)}(\mathbf{z}) \\ |\Delta_{xy}(\mathbf{z})| \geq 1}} d(\Delta_{xy}(\mathbf{z})) \left(\frac{q \sqrt{X_1 X_2 Y_1 Y_2}}{\sqrt{|\Delta_{xy}(\mathbf{z})|}} + 1 \right).$$

Next, we process the condition that $q \mid L_z^{(j)}(\mathbf{z})$ for all j . Let C be the 4×2 matrix having the $L_z^{(j)}$ as rows. Since f is irreducible, there exists no prime p that divides all elements of C . Since $q \mid L_z^{(j)}(\mathbf{z})$, we have as in the proof of Lemma 10 that q divides all 2×2 minors of C . Therefore, we may apply Lemma 3 and deduce that the elements \mathbf{z} counted by the above inner sum are in a lattice Λ_q of determinant q . We further observe that the points \mathbf{z} counted by the inner sum are also in the ellipse $E(Z_1, Z_2)$ given by $(z_1/Z_1)^2 + (z_2/Z_2)^2 \ll 1$. This ellipse has area $A(E(Z_1, Z_2)) \asymp Z_1 Z_2$. By Lemma 4, there exists a basis $\mathbf{b}_1, \mathbf{b}_2$ of the lattice Λ_q and a positive number α such that if we write $\mathbf{z} \in E(Z_1, Z_2)$ as $\mathbf{z} = G\mathbf{z}'$ then $|z'_1| \leq \alpha$ and $|z'_2| \ll Z_1 Z_2 / (\alpha q)$. Here, G is a 2×2 matrix with columns \mathbf{b}_1 and \mathbf{b}_2 . Note that Δ_{xy} is a quadratic form of discriminant D and thus, the discriminant of the new quadratic form $\Delta'_{xy}(\mathbf{z}') = \Delta_{xy}(G\mathbf{z}')$ is $q^2 D$. We further note that if $\Delta'_{xy}(\mathbf{z}') = 0$ then $\Delta_{xy}(\mathbf{z}) = 0$, and thus it is safe to assume that $|\Delta'_{xy}(\mathbf{z}')| \geq 1$. By following the proof of Lemma 4, we can see that

$$\|\Delta'_{xy}\| \ll \|\Delta_{xy}\| P(D, Z_1, Z_2),$$

where $P(D, Z_1, Z_2)$ is a finite power of $DZ_1 Z_2$. Thus, after a change of variables $\mathbf{z} = G_q \mathbf{z}'$ for each q , we arrive at the estimate

$$\mathcal{N} \ll \sum_{q^2 \mid D} \sum_{\substack{\mathbf{z}': |z'_i| \leq Z'_i \\ (z'_1, z'_2) = 1 \\ |\Delta'_{xy}(\mathbf{z}')| \geq 1}} d(\Delta'_{xy}(\mathbf{z}')) \left(\frac{q \sqrt{X_1 X_2 Y_1 Y_2}}{\sqrt{|\Delta'_{xy}(\mathbf{z}')|}} + 1 \right),$$

where $Z'_1 Z'_2 \ll Z_1 Z_2 / q$. If $Z'_1 < 1$ then $z'_1 = 0$ and $z'_2 = \pm 1$. A similar argument holds if $Z'_2 < 1$. Thus, we note that

$$1 \leq |\Delta'_{xy}(\mathbf{z}')| \ll \|\Delta_{xy}\| P(D, Z_1, Z_2),$$

where again $P(D, Z_1, Z_2)$ denotes some finite power of $DZ_1 Z_2$. We may therefore deduce the trivial estimate

$$\sum_{q^2 | D} \sum_{\substack{\mathbf{z}': |z'_i| \leq Z'_i \\ (z'_1, z'_2) = 1 \\ |\Delta'_{xy}(\mathbf{z}')| \geq 1}} d(\Delta'_{xy}(\mathbf{z}')) \ll T^\epsilon Z_1 Z_2$$

and it remains to find an upper bound for the remaining sum. Thus, we may split the sum over \mathbf{z}' into dyadic ranges for $|\Delta'_{xy}(\mathbf{z}')|$. In particular, there exists an integer R such that $1 \leq R \ll \|\Delta_{xy}\| P(D, Z_1, Z_2)$ and

$$\mathcal{N} \ll T^\epsilon \sqrt{X_1 X_2 Y_1 Y_2} \sum_{q^2 | D} q \sum_{\substack{\mathbf{z}': |z'_i| \leq Z'_i \\ (z'_1, z'_2) = 1 \\ R \leq |\Delta'_{xy}(\mathbf{z}')| < 2R}} \frac{1}{\sqrt{|\Delta'_{xy}(\mathbf{z}')|}} + T^\epsilon Z_1 Z_2$$

We proceed to attack the inner sum. For $q, n > 0$ with $q^2 \mid D$, let

$$a_{n,q} := \# \{ \mathbf{z}' \in \mathbb{Z}^2 : |z'_i| \leq Z'_i, (z'_1, z'_2) = 1, |\Delta_{xy}(\mathbf{z}')| = n \}.$$

Then

$$\mathcal{N} \ll T^\epsilon \sqrt{X_1 X_2 Y_1 Y_2} \sum_{q^2 | D} q S_q(R) + T^\epsilon Z_1 Z_2,$$

where

$$S_q(R) = \sum_{R \leq n < 2R} \frac{a_{n,q}}{\sqrt{n}}.$$

We now need to find an upper bound for $S_q(R)$. For $t \geq 1$, let

$$A_q(t) := \sum_{1 \leq m < 2t} a_{m,q}.$$

By partial summation, we obtain

$$S_q(R) \ll \frac{A_q(2R)}{\sqrt{R}} + \frac{A_q(R)}{\sqrt{R}} + \int_R^{2R} \frac{A_q(t)}{t^{3/2}} dt.$$

It remains to find an upper bound for $A_q(t)$. We prove the following lemma:

Lemma 13. *The following upper bound holds:*

$$A_q(t) \ll \min \left\{ \frac{Z_1 Z_2}{q} + 1, \left(\frac{t}{q D^{1/2}} + 1 \right) (Tt)^\epsilon \right\}$$

Proof. Note that

$$A_q(t) = \# \{ \mathbf{z} \in \mathbb{Z}^2 : |z_i| \leq Z'_i, (z_1, z_2) = 1, 1 \leq |\Delta'_{xy}(\mathbf{z})| \leq t \},$$

where we recall that Z'_1 and Z'_2 are positive numbers such that $Z'_1 Z'_2 \ll Z_1 Z_2 / q$ and Δ'_{xy} is a quadratic form with integer coefficients such that $D(\Delta'_{xy}) = q^2 D$. If $Z'_1 < 1$ then $z_1 = 0$ and $z_2 = \pm 1$. Thus, the case when $Z'_1 < 1$ or $Z'_2 < 1$ contributes $O(1)$ to $A_q(t)$. If $Z'_1 \geq 1$ and $Z'_2 \geq 1$ then

$$A_q(t) \ll (Z'_1 + 1)(Z'_2 + 1) \ll \frac{Z_1 Z_2}{q} + 1.$$

This proves the first bound. Next, let us consider the case when

$$\Delta'_{xy}(\mathbf{z}) = \alpha(z_1 - \beta_1 z_2)(z_1 - \beta_2 z_2) = \alpha L_1(\mathbf{z}) L_2(\mathbf{z}),$$

say, for some $\alpha \in \mathbb{Z} \setminus \{0\}$ and $\beta_i \in \mathbb{C}$. Note that in general $D(\Delta'_{xy}) = |\alpha|^2 |\beta_1 - \beta_2|^2$. Such a factorization exists if and only if the coefficient of z_1^2 in Δ' is non-zero. We first consider the case when $\beta_1 = \beta_2$. In this case we can deduce that $q^2 D = D(\Delta'_{xy}) = 0$, which is impossible since $D \neq 0$.

Next consider the case when $\beta_1 = \bar{\beta}_2 = a + bi$, with $a, b \in \mathbb{R}$ and $b \neq 0$. Then

$$|\Delta'_{xy}(\mathbf{z})| = |\alpha|((x - ay)^2 + (by)^2).$$

We set $r = x - ay$ and $s = by$. Then the point (r, s) is inside a real lattice Λ given by the matrix

$$\begin{pmatrix} 1 & -a \\ 0 & b \end{pmatrix}.$$

We note that $\det(\Lambda) = |b|$ and that the point (r, s) also lies in the circle given by

$$r^2 + s^2 \leq t/|\alpha|.$$

This circle has area $\pi t/|\alpha|$. Thus, by Lemma 4, the number of possibilities for (r, s) is

$$\ll \frac{t}{|\alpha||b|} + 1 \ll \frac{t}{qD^{1/2}} + 1,$$

where in the last line we used that $q^2 D = D(\Delta'_{xy}) = 4b^2 \alpha^2$. Each choice of (r, s) gives at most one choice for \mathbf{z} since $b \neq 0$.

We may therefore assume that β_1 and β_2 are real. By the solution formula for quadratic polynomials, we observe that $\beta_i \ll \|\Delta'\|/|\alpha|$ and therefore, $\alpha L_i(\mathbf{z}) \ll \|\Delta'\|Z$, where $Z = \max\{Z'_1, Z'_2\}$. Since $1 \leq |\Delta'_{xy}(\mathbf{z})|$, we also note that $|L_i(\mathbf{z})| \gg 1/(\|\Delta'\|Z)$. We therefore conclude that

$$\frac{1}{\|\Delta'\|Z} \ll |L_i(\mathbf{z})| \ll \|\Delta'\|Z.$$

We proceed by splitting the ranges of $L_1(\mathbf{z})$ and $L_2(\mathbf{z})$ into dyadic ranges so that

$$K \leq |L_1(\mathbf{z})| < 2K \text{ and } L \leq |L_2(\mathbf{z})| < 2L$$

for some integers K and L . The dyadic subdivision comes at the cost of a factor $(\|\Delta_{xy}\| D Z_1 Z_2 t)^\epsilon$ in the estimate for $A_q(t)$. By setting $r = L_1(\mathbf{z})$ and $s = L_2(\mathbf{z})$ we can again see that the points (r, s) are in a real lattice Λ with $|\det(\Lambda)| = |\beta_1 - \beta_2|$ and they are in an ellipse of shape

$$(r/K)^2 + (s/L)^2 \ll 1$$

and area $\ll KL \ll \frac{t}{|\alpha|}$. And thus, we obtain once again by Lemma 4 that the number of possibilities for the points (r, s) is

$$\ll (\|\Delta_{xy}\| D Z_1 Z_2 t)^\epsilon (t/(qD^{1/2}) + 1).$$

The linear transformation defining (r, s) is invertible because $\beta_1 \neq \beta_2$. Thus, for each choice (r, s) there is again at most one possible value for \mathbf{z} . This finishes the proof of the lemma if the coefficient of z_1^2 in Δ' is non-zero. The case when the coefficient of z_2^2 is non-zero is similar.

Therefore, we may now concentrate on the last case when $\Delta'_{xy}(\mathbf{z}) = bz_1z_2$ for some integer $b \neq 0$. We recall that $D(\Delta'_{xy}) = q^2D = b^2$. In particular, $b = qD^{1/2}$. We observe that

$$A_q(t) \ll \# \{ \mathbf{z} \in \mathbb{Z}^2 : 1 \leq |bz_1z_2| \leq t \} \ll t^\epsilon \left(\frac{t}{b} + 1 \right) \ll t^\epsilon \left(\frac{t}{qD^{1/2}} + 1 \right).$$

□

We note that $q < \sqrt{R}$ because $q^2 \mid \Delta_{xy} \ll R$. This gives

$$S_q(R) \ll (TR)^\epsilon \left(\frac{1}{q} \min \left\{ \frac{Z_1 Z_2}{\sqrt{R}}, \frac{\sqrt{R}}{D^{1/2}} \right\} + \frac{1}{\sqrt{R}} \right) \ll \frac{T^\epsilon}{q} \left(\frac{\sqrt{Z_1 Z_2}}{D^{1/4}} + 1 \right),$$

where the critical value is obtained when $R = Z_1 Z_2 D^{1/2}$. This proves the theorem.

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